Circuit Challenge 3

Consider **Circuit 1** (a series-resonant configuration):

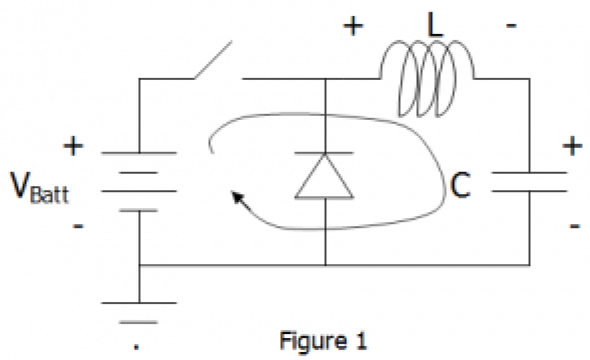

Figure 1

Initial Conditions

- The switch has been open for a long time.
- There is no charge on the capacitor; therefore its voltage is 0V.
- The voltage on the capacitor is not changing; therefore there is no current in the capacitor.
- No current is flowing in the loop; therefore there is no energy in the inductor.
- The current is not changing in the loop; therefore there is no voltage on the inductor.

To summarize: This circuit is at rest. Except for the battery, not one scintilla of energy can be found in the network.

Circuit Elements

- The battery is ideal (no resistance, no inductance, and the voltage is invariant with load)
- The switch is ideal (∞ Ohms when open, 0 Ohms when closed, no inductance, no bounce)
- The diode is ideal (no forward voltage drop, infinite reverse breakdown, no capacitance)
- The inductor is ideal (no resistance, no capacitance, and L is a fixed value)
- The capacitor is ideal (no resistance, no inductance, and C is a fixed value)
- The wire is ideal (no resistance, no inductance)

The Challenge

At time t=0 the switch is closed, and it remains closed. You are to answer the following questions**:**

1. What is the maximum voltage that will appear on the capacitor?
2. When (t=?) will this maximum voltage appear?
3. What maximum current will flow in this circuit?
4. When (t=?) does this maximum current flow?

You may verify your results by scrolling down to the analysis below.

Analysis for Circuit Challenge 3

This is a resonant circuit, and will therefore exhibit sinusoidal behavior. Immediately after the switch closes the following will be true (using the L & C relationships with respect to time):

$$V_{Batt} = Ldi/dt + 1/C\int idt$$

If this equation is differentiated with respect to time, we get

$$0 = Ld^2i/dt^2 + i/C$$

If we adopt a more streamlined way of presenting this, we have

$$0 = Li'' + i/C$$

Dividing through by L, and re-ordering things yields

$$i'' + i/(LC) = 0$$

which is a 2nd order differential equation with the following solution:

$$i = A\sin\omega t + B\cos\omega t$$

where

$$\omega = 1/SQRT(LC)$$

i=0 at t=0 because current cannot change instantaneously in an inductor, and because the diode is perfect (it will not break down, and it will permit no current to flow in reverse through it). And, because cos0=1 and sin0=0 we can write

$$0 = Ax0 + Bx1$$

We conclude that B=0, so that we now have

$$i = A\sin\omega t$$

Immediately following switch closure we note that all of V_{Batt} will appear on L. So, we can write that

$$V_{Batt} = Li'$$

$$V_{Batt} = LA\omega\cos\omega t$$

Since cos0 = 1 we can write

$$V_{Batt} = LA\omega$$

Solving for A, we get

$$A = V_{Batt}/(L\omega)$$

Substituting in for ω we solve for A, and get

$$A = V_{Batt}SQRT(C/L)$$

A is in amperes, as expected. So, we now have

$$i = [V_{Batt}SQRT(C/L)]\sin\omega t$$

Because we assumed there is no "bounce" in the switch the diode is completely out of the picture. In practical circuits it is present to provide a current path if the switch *does* open (because current cannot change instantaneously in an inductor, and the inductor *will* force current to go somewhere).

We now seek to determine the maximum voltage on the capacitor, so we write

$$v_C = (1/C)\int idt$$

$$v_C = (1/C)\int [V_{Batt}SQRT(C/L)](\sin\omega t)dt$$

$$v_C = (1/C)[V_{Batt}SQRT(C/L)](\sin\omega t)dt$$

$$v_C = (1/C)[V_{Batt}SQRT(C/L)](1/\omega)(-\cos\omega t + k)$$

Because voltage cannot change instantaneously on a capacitor we know that $V_C=0$ at $t=0$. Inasmuch as $\cos 0=1$, it is clear that $k = 1$. So, we can now write

$$v_C = (1/C)[V_{Batt}SQRT(C/L)](1/\omega)(1-\cos\omega t)$$

Certainly this is 0 at $t=0$, but what is v_C when $\omega t=\pi$? At $\omega t=\pi$ we will have the maximum, because

$$1-\cos\pi = 1-(-1)$$

$$= 2$$

If we grind through the prefix of $(1-\cos\omega t)$ to see if we can simplify the expression we will get (voila!)

$$(1/C)[V_{Batt}SQRT(C/L)](1/\omega) = V_{Batt}$$

So, we can write

$$v_C = V_{Batt}(1-\cos\omega t)$$

And we conclude the following:

- The peak voltage on the capacitor will be $2V_{BATT}$ (the circuit is a voltage doubler)
- The peak voltage will occur at $\omega t = \pi, 3\pi, 5\pi, \ldots$
- The maximum current will be V_{Batt}SQRT(C/L), +or –, depending on t
- The maximum current occurs at $\omega t = \pi/2, 3\pi/2, 5\pi/2, \ldots$

Observation 1

If the actual switch used in **Figure 1** is a relay (rather than a MOSFET), there is a danger if the switch opens while current is negative (i.e., flowing from right-to-left in the inductor). Current cannot change instantaneously in an inductor. If the switch opens, and if there is no path into which this reverse current can easily flow, the inductor will *force* a path to exist (via arcing, and/or reverse breakdown of the diode that is now drawn in the circuit schematic).

Observation 2

If the actual switch used in **Figure 1** is a MOSFET, a circuit designer usually has a built-in (but not usually drawn) safety valve: A bulk (body) diode that will provide a low-impedance path to the battery in the event of a surprise reverse-current event. For those who want to know more, please see Reference 1 or Reference 2.

Want to Simulate This Challenge, But You Don't Have a Simulator?

Try this one —> http://www.5spice.com

The More General Case

Although closing the switch when $v_C=0$, and when $i_{Loop}=0$ is an important case (i.e., a start-up for a switching regulator), two (likely simultaneous) initial conditions should also be considered. Those initial conditions are:

- A pre-existing loop current, I_0 (flowing in L, C, and through the diode)
- A pre-existing voltage on the capacitor, V_{C0}

Recall that a regulated system has the switch opening & closing. We now define t=0 to be the moment just after the n^{th} closure of the switch. We assume again that some anomalous condition removes the switch from its usual controlled state of operation. That is, once closed it cannot re-open. Then we look again at Figure 1. We notice first that the diode is removed from consideration just as soon as it is forced to be in parallel with the battery. Current in the inductor cannot change instantaneously; therefore I_0 continues to flow in L & C, but is diverted entirely to the battery. If we follow the original analysis it becomes apparent that we should now step in where, as before,

$$i = A\sin\omega t + B\cos\omega t$$

If we set t equal to 0 it is clear that $B=I_0$ (because the sine of 0 is 0, and the cosine of 0 is 1)

We note (again) that

$$v_L = L\,di/dt$$

Then

$$v_L = LA\omega\cos\omega t – B\omega\sin\omega t$$

Noting that $v_C(t=0)$ equals V_{C0} (the defined initial condition), we can write

$$v_L(t=0) = V_{Batt}-V_{C0}$$

And, because the cosine of 0 is 1, and the sine of 0 is 0, we can write

$$V_{Batt}-V_{C0} = LA\omega$$

A is now perfectly defined.

$$A = (V_{Batt}-V_{C0})/L\omega$$

Substituting 1/SQRT(LC) for ω permits A to be written even more precisely as

$$A = (V_{Batt}-V_{C0})SQRT(C/L)$$

So, we now have a complete solution for i using only the applied voltage, circuit values, and the given initial conditions.

$$i = (V_{Batt}-V_{C0})SQRT(C/L)\sin\omega t + I_0\cos\omega t$$

As before, the voltage on the inductor is given by Ldi/dt. So, we can write

$$v_L = L(V_{Batt}-V_{C0})SQRT(C/L)\omega\cos\omega t – I_0\omega\sin\omega t$$

Expanding ω to 1/SQRT(LC), we can simplify v_L to the following form

$$v_L = (V_{Batt}-V_{C0})\cos\omega t – I_0SQRT(L/C)\sin\omega t$$

At this point we express v_C as the battery voltage less the inductor voltage, as follows

$$v_C = V_{Batt} – (V_{Batt}-V_{C0})\cos\omega t + I_0SQRT(L/C)\sin\omega t$$

Rearranging gives

$$v_C = V_{Batt}(1-\cos\omega t) + V_{C0}\cos\omega t + I_0SQRT(L/C)\sin\omega t$$

If (as a sanity check) we set V_{C0} to 0 and I_0 to 0 we get precisely the same expression for v_C that derives from the start-up case (as we should). Or, if (as a second sanity check) we set V_{C0} to V_{Batt} and I_0 to 0 we get a circuit condition that will remain at rest with that passage of time (no voltage changes, no current changes), as we should expect. So, we now have a general solution summary for all cases.

Solution Summary for the General Case of Non-zero Initial Conditions

$$i_{Loop} = (V_{Batt}-V_{C0})(SQRT(C/L))\sin\omega t + I_0\cos\omega t$$

$$v_L = (V_{Batt}-V_{C0})\cos\omega t - I_0 SQRT(L/C)\sin\omega t$$

$$v_C = V_{Batt}(1-\cos\omega t) + V_{C0}\cos\omega t + I_0 SQRT(L/C)\sin\omega t$$

where

$$\omega = 1/SQRT(LC)$$

Circuit 2 (a parallel-resonant configuration)

Three ideal circuit elements in parallel

- A current source ($i_{Source} = 0$ for all $t < 0$, $i_{Source} = I$ for all $t \geq 0$)
- An inductor, L
- A capacitor, C

Initial Conditions

- The charge on the capacitor is 0. Therefore its voltage is $v_C(0)=0V$
- No current is flowing anywhere in the network, nor is any current changing

The Circuit Elements

- The current source is ideal (infinite impedance, no capacitance, and the current is invariant with voltage)
- The inductor is ideal (no resistance, no capacitance, and L is a fixed value)
- The capacitor is ideal (no resistance, no inductance, and C is a fixed value)
- The wire is ideal (no resistance, no inductance)

The Event

At t=0 the current source steps from 0A to I, where I is a constant.

The Tasks

- Determine an expression for the voltage across the network.
- Determine an expression for current in the capacitor.
- Determine an expression for current in the inductor.

Observations

1. The voltage will be 0V just after the current steps to I because voltage cannot change instantaneously on a capacitor.
2. $i_L = 0$ just after the current steps to I because current cannot change instantaneously in an inductor.
3. $i_C = I$ just after the current steps to I because no current is flowing in the inductor.
4. Even though the *magnitude* of v_C at t=0 is 0V, the *slope* of v_C at t=0 must be >0 because $i_C=C(dv/dt)$, and i_C is I at t=0.

The Analytic Solution

Kirkhoff's current law must be satisfied (the sum of the currents into any node must be zero). Thus,

$$I - i_L - i_C = 0$$

Recognizing that i_C=Cdv/dt, and then re-writing, we get

$$I - i_L - Cdv/dt = 0$$

Differentiating with respect to time, we have

$$0 - di_L/dt - Cv'' = 0$$

But Ldi_L/dt is the voltage on the inductor, which is the same as the voltage on the capacitor, which is v. So, we can write

$$0 - v/L - Cv'' = 0$$

Multiplying through by -1, dividing by C, and then rearranging, gives

$$v'' + v/(LC) = 0$$

In form, this is identical to the switched-voltage case. Therefore, the solutions for v, i_C, and i_L will be sinusoidal, and the analysis proceeds in a manner similar to the series-resonant case. Without going through it step-by-step, the solution summary is given below.

Solution Summary for the Case of Zero-volt and Zero-current Initial Conditions

$$v_{CurrentSource} = v_L = v_C = I[SQRT(L/C)]\sin\omega t$$

$$i_C = I\cos\omega t$$

$$i_L = I(1-\cos\omega t)$$

where

$$\omega = 1/SQRT(LC)$$

Implications

The current peak will be 2I in the inductor. This circuit may therefore be referred to as a current doubler. For that to predictably occur the current source must not saturate (if structured from a bipolar device) or go into the linear region of operation (if structured from a MOSFET). Therefore the peak voltage must be controlled by managing I, L, and C.

When Resistance Is a Factor

Series RLC Circuitry

With the possible exception of circuit operation near Absolute Zero (0 °K), there are no known zero-resistance circuits. The switch (usually a MOSFET) and the inductor would be likely candidates for introducing enough series resistance to merit some investigation. An analytical treatment of series RLC circuitry begins by placing a series R in Figure 1, followed by writing the following differential equation for the circuit, after the switch has closed:

$$V_{Batt} = iR + Ldi/dt + 1/C\int idt$$

The analysis is more complex, and there are three solutions (depending on the relative values of R, L, and C). We will start it here by differentiating the above equation with respect to t. Using the abbreviated notation for the derivatives, we get

$$0 = Ri' + Li'' + i/C$$

Rearranging, we get

$$Li'' + Ri' + i/C = 0$$

As stated, there are three solutions, and they derive from the following relationships:

If $R^2 - 4L/C > 0$, then

$$i = e^{-[R/(2L)]t}\{A_1e^{[+SQRT(R*R-4L/C)/(2L)]t} + B_1e^{[-SQRT(R*R-4L/C)/(2L)]t}\}$$

If $R^2 - 4L/C = 0$, then

$$i = e^{-[R/(2L)]t}(A_2 + B_2t)$$

If $R^2 - 4L/C < 0$, then

$$i = e^{-[R/(2L)]t}(A_3\cos\theta t + B_3\sin\theta t)$$

where θ is given by

$$\theta = [SQRT(|R^2-4L/C|)]/2L$$

and where, in the limit, as R becomes vanishingly small, θ approaches $\omega=1/SQRT(LC)$, as expected.

Determining A and B

An A and a B appear in each of the three general solutions. Determining A and B is essential. This is accomplished by first recognizing two initial conditions:

1. Immediately after the switch is thrown i=0 because current cannot change instantaneously in an inductor. Therefore i=0 at t=0.
2. Immediately after the switch is thrown $v_L=V_{Batt}$ because $v_C=0$ (a declared initial condition and voltage cannot change instantaneously on a capacitor) and $v_R=0$ (because $i_{Loop}=0$ at t=0). Thus, $Li'=V_{Batt}$ at t=0.

1 & 2 will yield two equations in two unknowns (always manageable).

The General Solutions

$$v_R = iR$$
$$v_L = Li'$$
$$v_C = 1/C\int i\,dt$$

where the applicable i is used (as determined by the value of R^2-4L/C). In the case of v_C, an alternative approach (if the integration appears challenging) would be to recognize that

$$v_C = V_{Batt} - iR - Li'$$

Parallel RLC Circuitry

Here we have four ideal circuit elements in parallel:

- A **current source** ($i_{Source} = 0$ for all $t < 0$, $i_{Source} = I$ for all $t \geq 0$)
- An **inductor**, L
- A **capacitor**, C
- A **resistor**, R

Initial Conditions:

- The charge on the capacitor is 0. Therefore its voltage, v_{C0}, is 0V.
- No current is flowing anywhere in the network, nor is any current changing.

The Analysis

Applying Kirkhoff's current law to one of the two nodes shows that

$$I - v/R - i_L - Cdv/dt = 0$$

And differentiating yields

$$0 - v'/R - i_L' - Cv'' = 0$$

Recognizing that $i_L' = v/L$ results in

$$0 - v'/R - v/L - Cv'' = 0$$

Multiplying through by -1 and rearranging gives

$$Cv'' + v'/R + v/L = 0$$

which is another classic second order differential equation. And we have three general solutions, determined by the values of C, R, and L, as follows:

If $1/R^2-4C/L > 0$, then

$$v = e^{-t/(2RC)}(A_1e^{\{+SQRT[1/(R*R)-4C/L]/(2C)\}t} + B_1e^{\{-SQRT[1/(R*R)-4C/L]/(2C)\}t})$$

If $1/R^2-4C/L = 0$, then

$$v = e^{-t/(2RC)}(A_2 + B_2t)$$

If $1/R^2-4C/L < 0$, then

$$v = e^{-t/(2RC)}(A_3\cos\theta t + B_3\sin\theta t)$$

where θ is given by

$$\theta = SQRT(|1/R^2-4C/L|)/2C$$

and where, in the limit, as R approaches infinity, θ approaches $\omega=1/SQRT(LC)$, as expected.

Determining A and B

An A and a B appear in each of the three general solutions. Determining A and B is essential. This is accomplished by first recognizing two initial conditions:

1. v=0 at t=0, because voltage cannot change instantaneously across a capacitor.
2. Cdv/dt=I at t=0, because $i_R=0$ (voltage is 0) and because $i_L=0$ at t=0 (i_L had been 0 for t<0, and cannot change instantaneously).

1 & 2 will yield two equations in two unknowns (always manageable).

The General Solutions

$$i_R = v/R$$
$$i_L = (1/L)\int vdt$$
$$i_C = Cv'$$

where the applicable v is used, as determined by the value of $1/R^2-4C/L$.
In the case of i_L, an alternative approach (if the integration appears challenging) would be to recognize that

$$i_L = I - i_R - i_C$$

References

1. **Advanced Engineering Mathematics** [C. R. Wylie, Jr., original copyright 1951 (with subsequent editions)].

2. http://www.sosmath.com/tables/diffeq/diffeq.html
(scroll down to **Second Order Differential Equations**)

Final Thoughts

If the series RLC circuit is simulated, or if the parallel RLC circuit is simulated, a review of the three applicable analytic solutions (the equations) may be helpful in understanding why the resulting waveforms look the way they do. Or, if (for some reason) R, or L, or C is unknown, but there is some empirical (e.g., bench) data, then simulated waveforms may provide enough insight to permit an educated guess as to what the data means.